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ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS

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1. Introduction

As an interesting construction in topology, Borsuk and Ulam [3] introduced the n-th *symmetric product* of a metric space (X, d), denoted by $F_n(X)$. Namely $F_n(X)$ is the space of non-empty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H, i.e., $F_n(X) = \{A \subset X \mid 1 \leq |A| \leq n\}$ and $d_H(A, B) = \inf\{\epsilon \mid A \subset B_d(B, \epsilon) \text{ and } B \subset B_d(A, \epsilon)\} = \max\{d(a, B), d(b, A) \mid a \in A, b \in B\}$ for any $A, B \in F_n(X)$ (see [10, p.6]).

For the symmetric products of $\mathbb{R}$, it is known that $F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty)$ and $F_3(\mathbb{R}) \approx \mathbb{R}^3$ (see Section 3). It was proved in [3] that $F_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ (written $F_n(\mathbb{I}) \approx \mathbb{I}^n$) if and only if $1 \leq n \leq 3$, and that for $n \geq 4$, $F_n(\mathbb{I})$ can not be embedded into $\mathbb{R}^n$, where $\mathbb{I} = [0, 1]$ has the usual metric. Thus, for $n \geq 4$, $F_n(\mathbb{R}) \not\approx \mathbb{R}^n$. Molski [12] showed that $F_2(\mathbb{I}^2) \approx \mathbb{I}^4$, and that for $n \geq 3$ neither $F_n(\mathbb{I}^2)$ nor $F_2(\mathbb{I}^n)$ can be embedded into $\mathbb{R}^{2n}$. Thus, for $n \geq 3$, $F_n(\mathbb{R}^2) \not\approx \mathbb{R}^{2n}$ and $F_2(\mathbb{R}^n) \not\approx \mathbb{R}^{2n}$.

Turning toward the symmetric product $F_n(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$, Chinen and Koyama [9] prove that for $n \in \mathbb{N}$, both $F_{2n-1}(\mathbb{S}^1)$ and $F_{2n}(\mathbb{S}^1)$ have the same homotopy type of the $(2n-1)$-sphere $\mathbb{S}^{2n-1}$. In [7] Bott corrected Borsuk's statement [4] and showed that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. In [9], another proof of it is given.

For a metric space (X, d), we denote by $\mathrm{Isom}_d(X)$ ($\mathrm{Isom}(X)$ for short) the group of all isometries from X into itself, i.e., $\phi : X \to X \in \mathrm{Isom}_d(X)$ if ϕ is a bijection satisfying that $d(x, x') = d(\phi(x), \phi(x'))$ for any $x, x' \in X$. Let $n \in \mathbb{N}$. Every isometry $\phi : X \to X$ induces an isometry $\chi_{(n)}(\phi) : (F_n(X), d_H) \to (F_n(X), d_H)$ defined by $\chi_{(n)}(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, there exists a natural monomorphism $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$. It is clear that $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism if and only if $\chi_{(n)}$ is an epimorphism, i.e., for every $\Phi \in \mathrm{Isom}_{d_H}(F_n(X))$ there exists $\phi \in \mathrm{Isom}_d(X)$ such that $\Phi = \chi_{(n)}(\phi)$.

In this paper, it is of interest to know whether $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism for a metric space (X, d). Recently, Borovikova and Ibragimov [5] prove that $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$ and that $\chi_{(3)} : \mathrm{Isom}_d(\mathbb{R}) \to \mathrm{Isom}_{d_H}(F_3(\mathbb{R}))$ is an isomorphism, where $\mathbb{R}$ has the usual metric d. The following result is a generalization of the result above and the affirmative answer to [6, p.60, Conjecture 2.1].

Theorem 1.1. *Let $l \in \mathbb{N}$ and let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ with the usual metric d. Then $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism for each $n \in \mathbb{N}$.*

In Section 4, we give the main ideas of proof of Theorem 1.1. In Example 5.2 below, we present a compact metric space (X, d) such that $\chi_{(n)}(\mathrm{Isom}_d(X)) \neq \mathrm{Isom}_{d_H}(F_n(X))$ for all $n \geq 2$, i.e., $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is not an isomorphism. And, in Section 3, we provide another proof of [5, Theorem 6]. Its proof is based on the proof of [11, Lemma 2.4].

2. Preliminaries

Notation 2.1. Let denote the set of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let d be the usual metric on $\mathbb{R}^l$, i.e., $d(x,y) = \{\sum_{i=1}^{l}(x_i - y_i)^2\}^{1/2}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in \mathbb{R}^l$. Write $\mathbb{S}^l = \{x = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} \mid \sum_{i=1}^{l+1} x_i^2 = 1\}$ with the length metric d. Denote the identity map from X into itself by id_X.

Definition 2.2. Let (X, d) be a metric space, let $x \in X$, let Y, Z be subsets of X and let $\epsilon > 0$. Set $d(Y, Z) = \inf\{d(y, x) \mid y \in Y, z \in Z\}$, and $B_d(Y, \epsilon) = \{x \in X \mid d(x, Y) \leq \epsilon\}$. If $Y = \{y\}$, for simplicity of notation, we write $B_d(y, \epsilon) = B_d(Y, \epsilon)$ and $S_d(y, \epsilon) = S_d(Y, \epsilon)$.

For $n \in \mathbb{N}$, the n-th *symmetric product* of X is defined by

$$F_n(X) = \{A \subset X \mid 1 \leq |A| \leq n\},$$

where $|A|$ is the cardinality of A. Write $F_{(m)}(X) = \{A \in 2^X \mid |A| = m\}$ for each $m \in \mathbb{N}$. Let $\mathrm{Isom}(X, Y) = \{\phi \in \mathrm{Isom}(X) \mid \phi(y) = y \text{ for each } y \in Y\}$ for $Y \subset X$. Set $r(A) = \min\{\{1\} \cup \{d(a, a') \mid a, a' \in A, a \neq a'\}\}$ for each $A \in F_n(X)$.

3. A metric space is bi-Lipschitz equivalent to the symmetric product of $\mathbb{R}$

In this section, we give another proof of [5, Theorem 6] which is based on the proof of [11, Lemma 2.4].

Definition 3.1. Let $n \in \mathbb{N}$. Set $F_n^*(\mathbb{I}) = \{A \in F_n(\mathbb{I}) \mid 0, 1 \in A\}$. It is known that $F_2^*(\mathbb{I}) = \{\{0,1\}\}$, $F_3^*(\mathbb{I}) = \{\{0,t,1\} \mid 0 \leq t \leq 1\} \approx \mathbb{S}^1$, and, $F_4^*(\mathbb{I}) = \{\{0,s,t,1\} \mid 0 \leq s \leq t \leq 1\}$ is homeomorphic to the dance hat (see [16]). In general, $F_{2n}^*(\mathbb{I})$ is contractible but not collapsible, and $F_{2n+1}^*(\mathbb{I})$ has the same homotopy type of $\mathbb{S}^{2n+1}$. In [1], it is called the spaces $F_{2n}^*(\mathbb{I})$, $n \geq 2$, *higher dimensional dunce hats* (see [1]).

Definition 3.2 ([11]). Let (X, d) be a metric space with $\operatorname{diam} X \leq 2$. Set $\operatorname{Cone}^o(X) = X \times [0, \infty)/(X \times \{0\})$, is said to be the *open cone over* X, with the metric $d_C([(x_1, t_1)], [(x_2, t_2)]) = |t_1 - t_2| + \min\{t_1, t_2\} \cdot d(x_1, x_2)$.

Definition 3.3. Let $f : (X, d) \to (Y, d')$ be a map. The map f is said to be *Lipschitz* (*bi-Lipschitz*, respectively) if there exists $L > 0$ such that

$$d'(f(x_1), f(x_2)) \leq L\, d(x_1, x_2)$$

$$(L^{-1}\, d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L\, d(x_1, x_2), \text{ respectively})$$

for any $x_1, x_2 \in X$. (X, d) is said to be *bi-Lipschitz equivalent to* (Y, d') if there exists a surjective bi-Lipschitz map from (X, d) to (Y, d').

Theorem 3.4 ([11]). *Let $n \in \mathbb{N}$ with $n \geq 2$. Then $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times \operatorname{Cone}^o(F_n^*(\mathbb{I})), \rho)$, where $\rho = \sqrt{d^2 + (d_H)_C^2}$.*

Sketch of Proof. Let $Z = \{A \in F_n(\mathbb{R}) \mid \min A = 0\}$. For every $A \in Z$ there exists the unique $E \in F_n^*(\mathbb{I})$ such that $A = tE$, where $t = \max A$.

Step1: $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times Z, \rho_1)$, where $\rho_1 = \sqrt{d^2 + (d_H)^2}$. In fact, we can show the following.

Step1.1: A map $f : F_n(\mathbb{R}) \to \mathbb{R} \times Z : A \mapsto (\min A, A - \min A)$ is $\sqrt{5}$-Lipschitz.

Step1.2: A map $f^{-1} : \mathbb{R} \times Z \to F_n(\mathbb{R}) : (b, A) \mapsto A + b$ is 2-Lipschitz.

Step2: (Z, d_H) is bi-Lipschitz equivalent to $(\operatorname{Cone}^o(F_n^*(\mathbb{I})), (d_H)_C)$. In fact, we can show the following.

Step2.1: A map $g : Z \to \operatorname{Cone}^o(F_n^*(\mathbb{I})) : tE \mapsto [(E, t)]$ is 1-Lipschitz.

Step2.2: A map $g^{-1} : \operatorname{Cone}^o(F_n^*(\mathbb{I})) \to Z : [(E, t)] \mapsto tE$ is 3-Lipschitz.

By the above, $(\mathrm{id}_{\mathbb{R}} \times g) \circ f : F_n(\mathbb{R}) \to \mathbb{R} \times Z \to \mathbb{R} \times \operatorname{Cone}^o(F_n^*(\mathbb{I}))$ is a bi-Lipschitz equivalence. □

Corollary 3.5. *$(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$.*

Proof. By Definition 3.1, $F_2^*(\mathbb{I})$ is one point, thus $(\operatorname{Cone}^o(F_2^*(\mathbb{I})), (d_H)_C)$ is corresponding to $([0, \infty), d)$. By Theorem 3.4, $(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$. □

The following result is first proved in [5, Theorem 6]. We give another proof by use of Theorem 3.4.

Corollary 3.6 ([5]). *$(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$.*

Sketch of Proof. We note $F_3^*(\mathbb{I}) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx \mathbb{S}^1$.

Step1: We can show that $(\operatorname{Cone}^o(F_n^*(\mathbb{I})), (d_H)_C)$ is bi-Lipschitz equivalent to $(\operatorname{Cone}^o(\mathbb{S}^1), (d|_{\mathbb{S}^1})_C)$.

Step2: We can show that $(\mathbb{R}^2, d)$ is bi-Lipschitz equivalent to $(\operatorname{Cone}^o(\mathbb{S}^1), (d|_{\mathbb{S}^1})_C)$.

By Theorem 3.4, $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$. □

Remark 3.7. We note that $F_2(\mathbb{R}^2) \approx \mathbb{R}^4$. Indeed, we can define a homeomorphism $h : F_2(\mathbb{R}^2) \to \mathbb{R}^2 \times \mathrm{Cone}^o(\mathbb{S}^1/x \sim -x)(\approx \mathbb{R}^4)$ by

$$h(A) = \begin{cases} \left(m(A), \left[\left(\frac{2(A-m(A))}{\operatorname{diam} A}, \operatorname{diam} A\right)\right]\right) & \text{if } \operatorname{diam} A \neq 0 \\ (m(A), \text{the cone point}) & \text{if } \operatorname{diam} A = 0, \end{cases}$$

where $m(A) = a$ if $A = \{a\}$ and $m(A) = (a + a')/2$ if $A = \{a, a'\}$. In general, we see that $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \mathrm{Cone}^o(\mathbb{S}^{l-1}/x \sim -x)$ for each $l \in \mathbb{N}$.

4. Isometries

Lemma 4.1. *Let $n \in \mathbb{N}$ and let (X, d) be a metric space such that*

(1) $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$ *for each* $\Phi \in \mathrm{Isom}(F_n(X))$*, and that*
(2) $\mathrm{Isom}(F_n(X), F_1(X)) = \{\mathrm{id}_{F_n(X)}\}$.

Then, $\chi_{(n)} : \mathrm{Isom}(X) \to \mathrm{Isom}(F_n(X))$ *is an isomorphism.*

Proof. Let $\Phi \in \mathrm{Isom}(F_n(X))$ and let $A_x = \{x\} \in F_1(X)$ for each $x \in X$. By assumption, $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$. Denote $\Phi(A_x) \in F_1(X)$ by $\{\phi(x)\}$ for each $x \in X$. Then, $\phi : X \to X : x \mapsto \phi(x)$ is an isometry. Set $\Phi' = \chi_{(n)}(\phi^{-1}) \circ \Phi \in \mathrm{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = \mathrm{id}|_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi_{(n)}(\phi)|_{F_1(X)}$ and $\chi_{(n)}(\phi^{-1}) = \left(\chi_{(n)}(\phi)\right)^{-1}$. By assumption, we have that $\Phi' = \mathrm{id}_{F_n(X)}$, therefore, $\Phi = \chi_{(n)}(\phi)$, which completes the proof. □

Definition 4.2. Let (X, d) be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup\{k \in \mathbb{N} \mid A_1, \ldots, A_k \in S_{d_H}(A, \epsilon), d_H(A_i, A_j) = 2\epsilon\ (i \neq j)\} \in \mathbb{N} \cup \{\infty\}.$$

Lemma 4.3. *Let $l, n \in \mathbb{N}$, let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ and let $\Phi \in \mathrm{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$.*

Sketch of Proof. Let $n \in \mathbb{N}$ with $n \geq 2$.

Step1: Let $A = \{a_1\} \in F_1(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)$. We can show that $D_n(A, \epsilon) = 3$.

Step2: Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \ldots, a_m\} \in F_{(m)}(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We can show that $D_n(A, \epsilon) > 3$.

Let $\Phi \in \mathrm{Isom}(F_n(X))$ and let $A \in F_n(X)$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$ for each $0 < \epsilon < \min\{r(A), r(\Phi(A))\}$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$. □

Lemma 4.4. *Let $l, n \in \mathbb{N}$. Then,* $\mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{\mathrm{id}_{F_n(\mathbb{R}^l)}\}$.

Sketch of Proof.
Step1: Let $l, n \in \mathbb{N}$ and let $\Phi \in \mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\Phi|_{F_2(\mathbb{R}^l)} = \mathrm{id}_{F_2(\mathbb{R}^l)}$.
Step2: Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\Phi \in \mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$ and let $A \in F_{(m)}(\mathbb{R}^l)$. We can show that $\Phi(A) \subset A$. If similar arguments apply to $\Phi(A)$ and Φ^{-1}, we obtain $A = \Phi^{-1}(\Phi(A)) \subset \Phi(A)$, therefore, $A = \Phi(A)$.
□

Lemma 4.5. *Let $l, n \in \mathbb{N}$. Then* $\mathrm{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{\mathrm{id}_{F_n(\mathbb{S}^l)}\}$.

Proof. Let $\Phi \in \mathrm{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l))$, $m \in \mathbb{N}$ with $2 \leq m \leq n$ and let $A \in F_{(m)}(\mathbb{S}^l)$. We show that $A = \Phi(A)$. Let $a \in A$ and let $a' \in \mathbb{S}^l$ be the anti-point of a. Since $d_H(\{a'\}, \Phi(A)) = d_H(\Phi(\{a'\}), \Phi(A)) = d_H(\{a'\}, A) = \pi$, we have $a \in \Phi(A)$, therefore, $A \subset \Phi(A)$. If similar arguments apply to $\Phi(A)$ and Φ^{-1}, we obtain $\Phi(A) \subset \Phi^{-1}(\Phi(A)) = A$, therefore, $A = \Phi(A)$, which completes the proof. □

The proof of Theorem 1.1. By Lemmas 4.3, 4.4 and 4.5, the conditions in Lemma 4.1 hold for (X, d), which completes the proof. □

5. Questions

Question 5.1. *Let $l, n \in \mathbb{N}$ with $n \geq 2$. When (X, d) is a following space, is $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ an isomorphism?*

1. *$X = \mathbb{R}^l$ has a metric d_∞, where $d_\infty(x, y) = \max\{|x_i - y_i| \mid i = 1, \ldots, l\}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in X$.*
2. *X is a convex subset of $\mathbb{R}^l$.*
3. *X is an $\mathbb{R}$-tree (see [2] for $\mathbb{R}$-trees).*
4. *X is the hyperbolic l-space (see [8] for the hyperbolic l-space).*

Example 5.2. Let $n, m \in \mathbb{N}$ with $2 \leq n \leq m$ and let (X, d) be an m-points discrete metric space satisfying that $d(x, x') = 1$ whenever $x \neq x'$. Then, $F_n(X)$ is a discrete metric space such that $d_H(A, A') = 1$ for any $A, A' \in F_n(X)$ with $A \neq A'$. Thus, $|\mathrm{Isom}(X)| = |X|! < |F_n(X)|! = |\mathrm{Isom}(F_n(X))|$, therefore, $\chi_{(n)} : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is not an isomorphism.

This drives us to the following question as the generalization of Theorem 1.1.

Question 5.3. *Let (X, d) be a connected metric space. Then, is $\chi_{(n)} : \mathrm{Isom}(X) \to \mathrm{Isom}(F_n(X))$ an isomorphism?*

Question 5.4. *It is known that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. Is $F_3(\mathbb{S}^1)$ bi-Lipschitz equivalent to $\mathbb{S}^3$?*

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